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Vibration analysis of rotating twisted and open conical shells

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Abstract

Based on a non-linear strain–displacement relationship of a non-rotating twisted and open conical shell on thin shell theory, a numerical method for free vibration of a rotating twisted and open conical shell is presented by the energy method, where the effect of rotation is considered as initial deformation and initial stress resultants which are obtained by the principle of virtual work for steady deformation due to rotation, then an energy equilibrium of equation for vibration of a twisted and open conical shell with the initial conditions is also given by the principle of virtual work. In the two numerical processes, the Rayleigh–Ritz procedure is used and the two in-plane and a transverse displacement functions are assumed to be algebraic polynomials in two elements. The effects of characteristic parameters with respect to rotation and geometry such as an angular velocity and a radius of rotating disc, a setting angle, a twist angle, curvature and a tapered ratio of cross-section on vibration performance of rotating twisted and open conical shells are studied by the present method.

Keywords: Non-linear strain–displacement relationship; Twisted and open conical shell; Principle of virtual work; Rayleigh–Ritz procedure

1. Introduction

Blades are often found in the aerospace, mechanical and automobile industries and others, such as turbofans and aerial propellers, it is very important for safety and life of machinery to determinate their dynamic characteristics accurately because they rotate at high speed. However it is a hard work and impossible to obtain dynamic solutions exactly as the result of their complex configurations such as twist, curvature, a non-uniform cross-section and a non-uniform thickness, which had attracted a lot of researchers' attention to the studies of computation models and analysis methods, such as beam and plate models, and Galerkin, Rayleigh–Ritz and finite element methods. Beam, which is a simple model of blades, was often used in study on the dynamics of blades in past (Rao, 1973, 1977, 1980) and is also used in engineering now under considering non-linear vibration, shear deformation, non-uniform cross-section, pre-twist, coupled vibration and concentrated mass (Hamdan and Al-Bedoor, 2001; Lin and Hsiao, 2001;

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Rao and Gupta, 2001; Surace et al., 1997; Yoo et al., 2001). Plate and shell are more approximate models of blades, but there was a little work done, such as rotating twisted plates by FEM using two different shape functions (Ramamurti and Kielb, 1984), rotating pre-twisted and tapered plates also by FEM of triangular shell elements with three nodes and eighteen degrees of freedom (Sreenivasamurthy and Ramamurti, 1981), shells with camber and twist on shallow shell theory by Ritz method (Leissa et al., 1982), rotating thin twisted plates by the Rayleigh–Ritz procedure (Tsuiji et al., 1995) and twisted cylindrical thin panels on general shell theory (Hu and Tsuiji, 1999), where the effects of parameters of the rotating systems on vibration characteristics were studied.

The main purpose in this paper is to develop an numerical analysis method for vibration of a rotating twisted and open conical shell based on the study of non-rotating twisted and open conical shells (Hu et al., submitted for publication). Considering a non-linear relationship between strains and displacements of the conical shells on thin shell theory and the influence of rotation, a numerical analysis of vibration is carried out by the energy method and the Rayleigh–Ritz procedure with algebraic polynomials in two elements as admissible displacement functions. There are two processes in present method, first, under the effect of rotation the deformation and the stress resultants in a twisted and open conical shell are solved by the principle of virtual work for steady deformation; second, an energy equilibrium of equation is presented by the principle of virtual work for vibration considering the initial quantities induced by a centrifugal force. The convergent property and the accuracy of the analysis method are studied. The effects of parameters of this system such as an angular velocity and a radius of a rotating disc, a setting angle, twist, curvature and a tapered ratio of cross-section on vibration characteristics of rotating twisted and open conical shells are investigated by the present method.

2. Theoretical analysis

A system constructed by a rotating disc with a radius x_0, which is assumed to be rigid, and a twisted and open conical shell fixed on the periphery of the disc at an orientation angle ϕ called a setting angle is shown in Fig. 1. Where a rectangular right handed Cartesian co-ordinate system is denoted by (X_1, Y_1, Z_1) which is fixed in space with O_0 at the center of the rotating disc and another right handed Cartesian co-ordinate system is depicted by $(x, y, \tilde{z})$ with a set of unit vectors $(\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ which rotates around the Z_1-axis at a constant angular velocity Ω. The x- and y-axes are taken in a lengthwise direction of the shell and a chordwise direction of an arc, respectively, the $\tilde{z}$-axis in a radial direction where an arc is divided from the middle, and the position of the origin O is defined by a distance e ($e = \beta' a$, β' is a constant and a is a radius of an arc on an arbitrary cross-section) apart from the point O_1. The configuration of the twisted and open conical shell is explained by the following, a_0 and a_1 are average radii of arcs at two ends of the conical shell and their corresponding lengths of the arcs are denoted by b_0 and b_1, respectively, β is a subtended angle

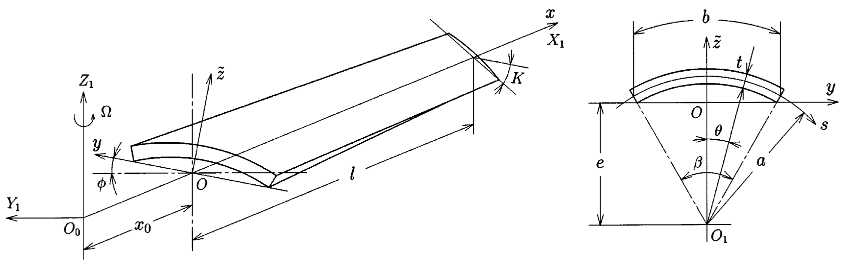

Fig. 1. A geometry of rotating twisted and open conical shell.

which is a constant for any cross-sections along the lengthwise direction, l and t represent a length and a thickness, respectively, and k is a twist ratio at which the conical shell is twisted round the x-axis and the twisted deformation is denoted by a twist angle K at an end of the conical shell defined as $K = kl$.

2.1. Strain under rotating effect

It is assumed that the displacement components of a point in a middle surface of the thin conical shell are u, v and w, respectively. To an arbitrary point which is in normal direction of the middle surface at a distance z, its displacement components $\overline{U}$, $\overline{V}$ and $\overline{W}$, and the displacement vector $\mathbf{\Gamma}$ can be written by (Hu et al., submitted for publication),

$$\begin{aligned}
\overline{U} &= u - \frac{z}{g}\left(\frac{p_{,x}}{\sqrt{g}}u + \frac{p_{,\theta}}{a\sqrt{g}}v + \frac{\partial w}{\partial x} - \frac{q}{a}\frac{\partial w}{\partial \theta}\right),\\
\overline{V} &= v - \frac{z}{\sqrt{g}}\left[\left(k - \frac{p_{,x}q}{g}\right)u - \frac{1}{a}\left(1 + \frac{p_{,\theta}q}{g}\right)v - \frac{q}{\sqrt{g}}\frac{\partial w}{\partial x} + \frac{\sqrt{g}}{a}\left(1 + \frac{q^2}{g}\right)\frac{\partial w}{\partial \theta}\right], \quad \overline{W} = w, \\
\Gamma &= \left(\overline{U} - \frac{p}{\sqrt{g}}\overline{W}\right)\boldsymbol{i}_1 + \left(f\overline{U} + \overline{V}\cos\theta + \frac{\sin\theta}{\sqrt{g}}\overline{W}\right)\boldsymbol{i}_2 + \left(h\overline{U} - \overline{V}\sin\theta + \frac{\cos\theta}{\sqrt{g}}\overline{W}\right)\boldsymbol{i}_3,
\end{aligned} \tag{1}$$

where the subscripts $(\ )_{,x}$ and $(\ )_{,\theta}$ denote partial differentiation with respect to x and θ, respectively, and the variables in the equations are defined in Appendix A. The engineering strains with respect to a local orthogonal co-ordinate system (ξ, η, ζ) consist of the linear and non-linear parts as given by Hu et al. (submitted for publication) or

$$\begin{Bmatrix} \varepsilon_{\xi\xi} \\ \varepsilon_{\eta\eta} \\ \gamma_{\xi\eta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^L_{\xi\xi} \\ \varepsilon^L_{\eta\eta} \\ \gamma^L_{\xi\eta} \end{Bmatrix} + \begin{Bmatrix} \varepsilon^N_{\xi\xi} \\ \varepsilon^N_{\eta\eta} \\ \gamma^N_{\xi\eta} \end{Bmatrix} = \frac{1}{1 - \bar{m}\frac{z}{l}}\boldsymbol{ZGU} + \frac{1}{2}\boldsymbol{U}^T \begin{Bmatrix} \boldsymbol{G}_x^T\boldsymbol{G}_x \\ \boldsymbol{G}_\theta^T\boldsymbol{G}_\theta \\ \boldsymbol{G}_x^T\boldsymbol{G}_\theta \end{Bmatrix}\boldsymbol{U}, \tag{2}$$

where the matrices $\boldsymbol{Z}$, $\boldsymbol{G}$ and $\boldsymbol{U}$ and $\boldsymbol{G}_x$ and $\boldsymbol{G}_\theta$ are

$$\begin{aligned}
\boldsymbol{Z} &= \begin{bmatrix} 1 & \frac{z}{l} & \frac{z^2}{l^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{z}{l} & \frac{z^2}{l^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{z}{l} & \frac{z^2}{l^2} \end{bmatrix}, \qquad \boldsymbol{G} = [G_{i,j}] \quad (i = 1\text{–}9;\ j = 1\text{–}12),\\
\boldsymbol{U}^T &= \left\{ \frac{\partial U}{\partial X} \quad \frac{\partial U}{\partial \theta} \quad U \quad \frac{\partial V}{\partial X} \quad \frac{\partial V}{\partial \theta} \quad V \quad \frac{\partial^2 W}{\partial X^2} \quad \frac{\partial^2 W}{\partial \theta^2} \quad \frac{\partial^2 W}{\partial X\partial \theta} \quad \frac{\partial W}{\partial X} \quad \frac{\partial W}{\partial \theta} \quad W \right\},\\
\boldsymbol{G}_x &= \left\{ 0 \quad 0 \quad \frac{\bar{p}_{,x}}{g} \quad 0 \quad 0 \quad \frac{p_{,\theta}}{\bar{a}g} \quad 0 \quad 0 \quad 0 \quad \frac{1}{\sqrt{g}} \quad -\frac{q}{\bar{a}\sqrt{g}} \quad 0 \right\},\\
\boldsymbol{G}_\theta &= \left\{ 0 \quad 0 \quad \frac{K}{\sqrt{g}} \quad 0 \quad 0 \quad -\frac{1}{\bar{a}\sqrt{g}} \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{\bar{a}} \quad 0 \right\},
\end{aligned} \tag{3}$$

and the non-zero elements in matrix $\boldsymbol{G}$ are defined in Appendix B where $\Phi_x^{(0)}$ and $\Phi_\theta^{(0)}$ are

$$\Phi_x^{(0)} = \frac{\bar{p}_{,x}}{g}U^{(0)} + \frac{p_{,\theta}}{\bar{a}g}V^{(0)} + \frac{1}{\sqrt{g}}\frac{\partial W^{(0)}}{\partial X} - \frac{q}{\bar{a}\sqrt{g}}\frac{\partial W^{(0)}}{\partial \theta}, \quad \Phi_\theta^{(0)} = \frac{K}{\sqrt{g}}U^{(0)} - \frac{1}{\bar{a}\sqrt{g}}V^{(0)} + \frac{1}{\bar{a}}\frac{\partial W^{(0)}}{\partial \theta}, \tag{4}$$

$U^{(0)}$, $V^{(0)}$ and $W^{(0)}$ are the initial deformation components produced by centrifugal forces.

All the quantities in above equations are dimensionless because the following dimensionless indices are introduced,

$$U=\frac{u}{l},\quad V=\frac{v}{l},\quad W=\frac{w}{l},\quad X=\frac{x}{l},\quad K=kl,\quad \bar{a}=\frac{a}{l},\quad \bar{e}=\frac{e}{l},\quad \bar{d}_2=d_2l,\quad \bar{d}_3=d_3,$$
$$\bar{m}=ml,\quad \bar{p}_{,x}=p_{,x}l,\quad \bar{q}_{,x}=q_{,x}l. \tag{5}$$

According to Eq. (2), the relationship between strains and stresses can be written as

$$\begin{Bmatrix}\sigma_{\xi\xi}\\ \sigma_{\eta\eta}\\ \tau_{\xi\eta}\end{Bmatrix}=\begin{bmatrix}\frac{E}{1-v^2} & \frac{vE}{1-v^2} & 0\\ \frac{vE}{1-v^2} & \frac{E}{1-v^2} & 0\\ 0 & 0 & \frac{E}{2(1+v)}\end{bmatrix}\begin{Bmatrix}\varepsilon^L_{\xi\xi}\\ \varepsilon^L_{\eta\eta}\\ \gamma^L_{\xi\eta}\end{Bmatrix}=\frac{1}{1-\bar{m}\frac{z}{l}}\boldsymbol{EZGU}, \tag{6}$$

where E and v indicate Young's modulus and Poisson ratio, respectively.

2.2. Initial deformation by centrifugal force

The thin conical shells are studied in this paper, it could be assumed that the centrifugal force is a constant throughout the thickness and Coriolis effect is neglected. Therefore, an arbitrary point in the deformed twisted and open conical shell can be expressed by a position vector $\boldsymbol{r}$ as follows:

$$\boldsymbol{r}=\left(\begin{Bmatrix}x_0+x\\ a\sin\theta\\ a\cos\theta-e\end{Bmatrix}^T+\begin{Bmatrix}u-\frac{p}{g}w\\ fu+v\cos\theta+\frac{\sin\theta}{\sqrt{g}}w\\ hu-v\sin\theta+\frac{\cos\theta}{\sqrt{g}}w\end{Bmatrix}^T\right)\begin{Bmatrix}\boldsymbol{i}_1\\ \boldsymbol{i}_2\\ \boldsymbol{i}_3\end{Bmatrix}. \tag{7}$$

The centrifugal force $\boldsymbol{F}$ per unit volume in the conical shell which is rotating at a constant angular velocity Ω is given by

$$\boldsymbol{F}=-\rho\frac{\partial^2\boldsymbol{r}}{\partial t^2_{\text{ime}}}=-\Omega^2\rho\left(\begin{Bmatrix}x_0+x\\ a\cos\theta\sin(\theta-\psi)+e\sin\theta\cos\theta\\ -a\sin\theta\sin(\theta-\psi)-e\sin^2\theta\end{Bmatrix}^T\right.$$
$$\left.+\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}^T\begin{bmatrix}1 & (f\cos\theta-h\sin\psi)\cos\psi & (h\sin\psi-f\cos\theta)\sin\psi\\ 0 & \cos\psi\cos(\theta-\psi) & -\sin\psi\cos(\theta-\psi)\\ -\frac{p}{\sqrt{g}} & \frac{\cos\psi}{\sqrt{g}}\sin(\theta-\psi) & -\frac{\sin\psi}{\sqrt{g}}\sin(\theta-\psi)\end{bmatrix}\right)\begin{Bmatrix}\boldsymbol{i}_1\\ \boldsymbol{i}_2\\ \boldsymbol{i}_3\end{Bmatrix}, \tag{8}$$

where $\psi=\phi+kx$.

For the twisted and open conical shell applied the force, the following energy equilibrium can be obtained by the principle of virtual work, namely,

$$0 = \iiint_{\mathrm{vol}} \left(\delta \left\{ \varepsilon^L_{\xi\xi} \quad \varepsilon^L_{\eta\eta} \quad \gamma^L_{\xi\eta} \right\} \left\{ \begin{matrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \tau_{\xi\eta} \end{matrix} \right\} + \delta \left\{ \varepsilon^N_{\xi\xi} \quad \varepsilon^N_{\eta\eta} \quad \gamma^N_{\xi\eta} \right\} \left\{ \begin{matrix} \sigma^{(0)}_{\xi\xi} \\ \sigma^{(0)}_{\eta\eta} \\ \tau^{(0)}_{\xi\eta} \end{matrix} \right\} \right) \sqrt{g}\left(1 - \bar{m}\frac{z}{l}\right) \mathrm{d}x\, a\, \mathrm{d}\theta\, \mathrm{d}z$$
$$+ \iiint_{\mathrm{vol}} \boldsymbol{F} \delta \boldsymbol{\Gamma} \sqrt{g}\left(1 - \bar{m}\frac{z}{l}\right) \mathrm{d}x\, a\, \mathrm{d}\theta\, \mathrm{d}z, \tag{9}$$

where $\sigma^{(0)}_{\xi\xi}$, $\sigma^{(0)}_{\eta\eta}$ and $\tau^{(0)}_{\xi\eta}$ denote the initial stresses produced by the centrifugal force, and ρ is a density of a material.

Substituting Eqs. (1), (2), (6) and (8) into Eq. (9), and integrating with respect to z, then readjusting the equation yield

$$\int\int_S \delta \boldsymbol{U}^T \left[\boldsymbol{G}^T \boldsymbol{D} \boldsymbol{G} + \boldsymbol{N}^{(0)}_{\xi\xi} \boldsymbol{G}^T_x \boldsymbol{G}_x + \boldsymbol{N}^{(0)}_{\eta\eta} \boldsymbol{G}^T_\theta \boldsymbol{G}_\theta + \boldsymbol{N}^{(0)}_{\xi\eta} (\boldsymbol{G}^T_x \boldsymbol{G}_\theta + \boldsymbol{G}^T_\theta \boldsymbol{G}_x) \right] \boldsymbol{U} \sqrt{g} a \, \mathrm{d}x \, \mathrm{d}\theta$$
$$- \Omega^2 \int\int_S \rho t [(\alpha_{11} u + \alpha_{12} v + \alpha_{13} w)\, \delta u + (\alpha_{21} u + \alpha_{22} v + \alpha_{23} w)\, \delta v + (\alpha_{31} u + \alpha_{32} v + \alpha_{33} w)\, \delta w] \sqrt{g} a \, \mathrm{d}x \, \mathrm{d}\theta$$
$$= \Omega^2 \int\int_S \rho t (\beta_1 u + \beta_2 v + \beta_3 w) \sqrt{g} a \, \mathrm{d}x \, \mathrm{d}\theta, \tag{10}$$

where the matrix $\boldsymbol{D}$ is defined as the following,

$$\boldsymbol{D} = \int_{-t/2}^{t/2} \boldsymbol{Z}^T \boldsymbol{E} \boldsymbol{Z} \frac{1}{1 - \bar{m}\frac{z}{l}} \mathrm{d}z = \frac{1}{l^2} \begin{bmatrix} \boldsymbol{D}_1 & v\boldsymbol{D}_1 & \boldsymbol{0} \\ & \boldsymbol{D}_1 & \boldsymbol{0} \\ \mathrm{Sym.} & & \frac{1-v}{2}\boldsymbol{D}_1 \end{bmatrix}, \quad \boldsymbol{D}_1 = \begin{bmatrix} Hl^2 & \bar{m}D & D \\ 0 & D & 0 \\ \mathrm{Sym.} & & 0 \end{bmatrix},$$
$$H = \frac{Et}{1 - v^2}, \quad D = \frac{Et^3}{12(1 - v^2)}, \tag{11}$$

the initial stress resultants are

$$\left\{ \begin{matrix} N^{(0)}_{\xi\xi} \\ N^{(0)}_{\eta\eta} \\ N^{(0)}_{\xi\eta} \end{matrix} \right\} = \int_{-t/2}^{t/2} \left\{ \begin{matrix} \sigma^{(0)}_{\xi\xi} \\ \sigma^{(0)}_{\eta\eta} \\ \tau^{(0)}_{\xi\eta} \end{matrix} \right\} \left(1 - \bar{m}\frac{z}{l}\right) \mathrm{d}z = \int_{-t/2}^{t/2} \boldsymbol{EZGU}^{(0)} \, \mathrm{d}z = \boldsymbol{D}_N \boldsymbol{GU}^{(0)}, \quad \boldsymbol{D}_N = \int_{-t/2}^{t/2} \boldsymbol{EZ} \, \mathrm{d}z, \tag{12}$$

and the indices in Eq. (10) are defined by

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} 1 & f & h \\ 0 & \cos\theta & -\sin\theta \\ -\frac{p}{\sqrt{g}} & \frac{\sin\theta}{\sqrt{g}} & \frac{\cos\theta}{\sqrt{g}} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix},$$
$$\beta_1 = f_1 + f f_2 + h f_3, \quad \beta_2 = f_2 \cos\theta - f_3 \sin\theta, \quad \beta_3 = \frac{1}{\sqrt{g}}(-p f_1 + f_2 \sin\theta + f_3 \cos\theta), \tag{13}$$

where

$$c_{11}=1,\quad c_{12}=0,\quad c_{13}=-\frac{p}{\sqrt{g}},\quad c_{21}=(f\cos\psi-h\sin\psi)\cos\psi,$$
$$c_{22}=\cos(\theta-\psi)\cos\psi,\quad c_{23}=\frac{1}{\sqrt{g}}\sin(\theta-\psi)\cos\psi,\quad c_{31}=(-f\cos\psi+h\sin\psi)\sin\psi,$$
$$c_{32}=-\cos(\theta-\psi)\sin\psi,\quad c_{33}=-\frac{1}{\sqrt{g}}\sin(\theta-\psi)\sin\psi,\quad f_1=x_0+x,$$
$$f_2=a\sin(\theta-\psi)\cos\psi+e\sin\psi\cos\psi,\quad f_3=-a\sin(\theta-\psi)\sin\psi+e\sin^2\psi. \tag{14}$$

For a demand on studying the effects of parameters of the system, the following non-dimensional form is produced by multiplying Eq. (10) with l^2/D, namely,

$$\begin{aligned}&\int\int_S \delta\boldsymbol{U}^T\left[\boldsymbol{G}^T\overline{\boldsymbol{D}}\boldsymbol{G}+\overline{N}^{(0)}_{\xi\xi}\boldsymbol{G}_x^T\boldsymbol{G}_x+\overline{N}^{(0)}_{\eta\eta}\boldsymbol{G}_\theta^T\boldsymbol{G}_\theta+\overline{N}^{(0)}_{\xi\eta}\left(\boldsymbol{G}_x^T\boldsymbol{G}_\theta+\boldsymbol{G}_\theta^T\boldsymbol{G}_x\right)\right]\boldsymbol{U}\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta\\&-\bar{\Omega}^2\lambda_0^2\int\int_S[(\alpha_{11}U+\alpha_{12}V+\alpha_{13}W)\,\delta U+(\alpha_{21}U+\alpha_{22}V+\alpha_{23}W)\,\delta V\\&+(\alpha_{31}U+\alpha_{32}V+\alpha_{33}W)\,\delta W]\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta=\bar{\Omega}^2\lambda_0^2\int\int_S\left(\bar{\beta}_1\delta U+\bar{\beta}_2\delta V+\bar{\beta}_3\delta W\right)\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta,\end{aligned} \tag{15}$$

where

$$\overline{\boldsymbol{D}}=\begin{bmatrix}\overline{\boldsymbol{D}}_1 & v\overline{\boldsymbol{D}}_1 & \boldsymbol{0}\\ & \overline{\boldsymbol{D}}_1 & \boldsymbol{0}\\ \text{Sym.} & & \frac{1-v}{2}\overline{\boldsymbol{D}}_1\end{bmatrix},\quad \overline{\boldsymbol{D}}_1=\begin{bmatrix}\bar{H} & \bar{m} & 1\\ & 1 & 0\\ \text{Sym.} & & 0\end{bmatrix},\quad \begin{Bmatrix}\overline{N}^{(0)}_{\xi\xi}\\ \overline{N}^{(0)}_{\eta\eta}\\ \overline{N}^{(0)}_{\xi\eta}\end{Bmatrix}^T=\frac{l^2}{D}\begin{Bmatrix}N^{(0)}_{\xi\xi}\\ N^{(0)}_{\eta\eta}\\ N^{(0)}_{\xi\eta}\end{Bmatrix}^T,$$
$$\bar{H}=\frac{Hl^2}{D},\quad \bar{\beta}_1=X_0+X,\quad \bar{\beta}_2=\bar{a}\cos(\theta-\psi)+\bar{e}\sin\psi\cos\psi,\quad \bar{\beta}_3=-\bar{a}\sin\psi\sin(\theta-\psi)-\bar{e}\sin^2\psi, \tag{16}$$

and the dimensionless parameter λ_0 is a reference frequency parameter of vibration.

With the purpose of the Rayleigh–Ritz procedure applied to the principle of virtual work in this paper, the dimensionless displacement components U, V and W are assumed as two-dimensional algebraic polynomial functions of X and θ, which satisfy the geometric boundary conditions employed at an end of the twisted and open conical shell ($X=0$) which is fixed to the periphery of the rotating disc, namely, $U=0$, $V=0$, $W=0$ and $\partial W/\partial X=0$, they are given by

$$U=\sum_{i=1}^{N_U}\sum_{j=0}^{M_U}a'_{ij}X^i\theta^j,\quad V=\sum_{k=1}^{N_V}\sum_{l=0}^{M_V}b'_{kl}X^k\theta^l,\quad W=\sum_{m=2}^{N_W}\sum_{n=0}^{M_W}c'_{mn}X^m\theta^n, \tag{17}$$

where a'_{ij}, b'_{kl} and c'_{mn} are unknown coefficients.

Substituting Eq. (17) into Eq. (15) and integrating over the surface area of the conical shell, then taking the variation of the equation with respect to coefficients a'_{ij}, b'_{kl} and c'_{mn} according to the Rayleigh–Ritz procedure yield the following governing equation,

$$\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13}\\ & \boldsymbol{A}_{22} & \boldsymbol{A}_{23}\\ \text{Sym.} & & \boldsymbol{A}_{33}\end{bmatrix}\begin{Bmatrix}\boldsymbol{a}'\\ \boldsymbol{b}'\\ \boldsymbol{c}'\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{P}_1\\ \boldsymbol{P}_2\\ \boldsymbol{P}_3\end{Bmatrix}, \tag{18}$$

where $\boldsymbol{A}_{ij}(i,j=1,2,3)$ are stiffness matrices, and $\boldsymbol{P}_i(i=1,2,3)$ are force vectors.

It is known that the unknown coefficients a'_{ij}, b'_{kl} and c'_{mn} can be solved from Eq. (18), therefore, the initial deformation and the initial stress resultants can be obtained by Eqs. (17) and (16).

2.3. Free vibration of rotating conical shells

Under the conditions of the initial deformation and the initial stress resultants, the principle of virtual work for free vibration of a twisted and open conical shell can be written as

$$\begin{aligned}
&\int\int\int_{\text{vol}}\left(\delta\left\{\varepsilon^L_{\xi\xi}\quad\varepsilon^L_{\eta\eta}\quad\gamma^L_{\xi\eta}\right\}\left\{\begin{matrix}\sigma_{\xi\xi}\\ \sigma_{\eta\eta}\\ \tau_{\xi\eta}\end{matrix}\right\}+\delta\left\{\varepsilon^N_{\xi\xi}\quad\varepsilon^N_{\eta\eta}\quad\gamma^N_{\xi\eta}\right\}\left\{\begin{matrix}\sigma^{(0)}_{\xi\xi}\\ \sigma^{(0)}_{\eta\eta}\\ \tau^{(0)}_{\xi\eta}\end{matrix}\right\}\right)\sqrt{g}\left(1-\bar{m}\frac{z}{l}\right)\mathrm{d}x\,a\,\mathrm{d}\theta\,\mathrm{d}z\\
&\quad-\Omega^2\int\int_S\rho t[(\alpha_{11}u+\alpha_{12}v+\alpha_{13}w)\,\delta u+(\alpha_{21}u+\alpha_{22}v+\alpha_{23}w)\,\delta v+(\alpha_{31}u+\alpha_{32}v+\alpha_{33}w)\,\delta w]\sqrt{g}a\,\mathrm{d}x\,\mathrm{d}\theta\\
&\quad-\int\int\int_{\text{vol}}\rho\omega^2\boldsymbol{\Gamma}\delta\boldsymbol{\Gamma}\sqrt{g}\left(1-\bar{m}\frac{z}{l}\right)\mathrm{d}x\,a\,\mathrm{d}\theta\,\mathrm{d}z=0,
\end{aligned}\tag{19}$$

where ω represents an angular frequency of vibration.

Substituting Eqs. (1), (2) and (6) into the above equation, integrating with respect to z and multiplying l^2/D yield a non-dimensional equation given as the following,

$$\begin{aligned}
&\int\int_S\delta\boldsymbol{U}^T\left[\boldsymbol{G}^T\overline{\boldsymbol{D}}\boldsymbol{G}+\overline{N}^{(0)}_{\xi\xi}\boldsymbol{G}^T_x\boldsymbol{G}_x+\overline{N}^{(0)}_{\eta\eta}\boldsymbol{G}^T_\theta\boldsymbol{G}_\theta+\overline{N}^{(0)}_{\xi\eta}\left(\boldsymbol{G}^T_x\boldsymbol{G}_\theta+\boldsymbol{G}^T_\theta\boldsymbol{G}_x\right)\right]\boldsymbol{U}\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta\\
&\quad-\bar{\Omega}^2\lambda_0^2\int\int_S[(\alpha_{11}U+\alpha_{12}V+\alpha_{13}W)\delta U+(\alpha_{21}U+\alpha_{22}V+\alpha_{23}W)\delta V+(\alpha_{31}U+\alpha_{32}V+\alpha_{33}W)\delta W]\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta\\
&\quad=-\lambda^2\int\int_S[(g+q^2)U\delta U+qU\delta V+qV\delta U+V\delta V+W\delta W]\sqrt{g}\bar{a}\,\mathrm{d}X\,\mathrm{d}\theta,
\end{aligned}\tag{20}$$

where dimensionless λ is a vibration frequency parameter defined by

$$\lambda^2=\frac{12(1-\nu^2)\rho\omega^2 l^4}{Et^2}.\tag{21}$$

By employing the Rayleigh–Ritz procedure with algebraic polynomials as displacement functions like Eq. (17), a set of $N_U\times(M_U+1)+N_V\times(M_V+1)+(N_W-1)\times(M_W+1)$ homogeneous equations are obtained from Eq. (20),

$$\begin{bmatrix}\boldsymbol{A}_{11}-\lambda^2\boldsymbol{B}_{11} & \boldsymbol{A}_{12}-\lambda^2\boldsymbol{B}_{12} & \boldsymbol{A}_{13}\\ & \boldsymbol{A}_{22}-\lambda^2\boldsymbol{B}_{22} & \boldsymbol{A}_{23}\\ \text{Sym.} & & \boldsymbol{A}_{33}-\lambda^2\boldsymbol{B}_{33}\end{bmatrix}\left\{\begin{matrix}\boldsymbol{a}'\\ \boldsymbol{b}'\\ \boldsymbol{c}'\end{matrix}\right\}=\boldsymbol{0}.\tag{22}$$

The requirement that the determinant of the set must vanish for a non-trivial solution provides another algebraic equation called the eigenfrequency equation, or

$$\begin{bmatrix}\boldsymbol{A}_{11}-\lambda^2\boldsymbol{B}_{11} & \boldsymbol{A}_{12}-\lambda^2\boldsymbol{B}_{12} & \boldsymbol{A}_{13}\\ & \boldsymbol{A}_{22}-\lambda^2\boldsymbol{B}_{22} & \boldsymbol{A}_{23}\\ \text{Sym.} & & \boldsymbol{A}_{33}-\lambda^2\boldsymbol{B}_{33}\end{bmatrix}=\boldsymbol{0},\tag{23}$$

where $\boldsymbol{B}_{ij}(i,j=1,2,3)$ denote mass matrices.

As aforementioned it is known that the numerical method includes solving the initial deformation and the initial stress resultants from Eq. (18), and solving the eigenvalues and eigenvectors of vibration from Eq. (23), which is an iterative procedure because Eq. (18) is a non-linear equation. Therefore, a criterion is necessary for the procedure control. Here, the frequency parameters λ are considered as a controlled quantity which satisfy the following equation,

$$\frac{\left|\lambda_i^{(j+1)} - \lambda_i^{(j)}\right|}{\lambda_i^{(j+1)}} \leqslant 0.01\% \quad (i = 1, \ldots, 10), \tag{24}$$

where the subscript i denotes the ith vibration mode and the superscript j denotes the jth iteration.

3. Numerical results and analyses

In the numerical analysis method for vibration of rotating thin conical shells, numerical integration is one of important operation. According to our knowledge and the experience of this kind of problem, Gauss–Legendre method with 16 integration points is adopted in the paper. Another important thing is the proper numbers of terms in the three displacement functions which is investigated in next. Poisson ratio v is 0.3.

3.1. Convergence of frequency parameters λ

An example of a rotating disc with a radius $X_0 = 2.0$ and an angular velocity $\bar{\Omega} = 1.0$, and a rotating twisted and open conical shell fixed on the disc with a setting angle $\phi = 45°$ is considered, where the geometric parameters of the conical shell are given as the following: a thickness ratio $b_0/t = 25$, an aspect ratio $b_0/l = 0.5$, a tapered ratio of cross-section $\alpha = 0.6$ defined as $\alpha = b_1/b_0$, a subtended angle $\beta = 60°$, a twist angle at a free end $K = 60°$, and the convergence against the numbers of the terms or the powers of the variables in the displacement functions is investigated. From Eq. (17) it is known that those displacements are two-dimensional functions and the numbers of terms in them are decided by the maximum powers of X and θ, respectively. There are no constraints for the maximum powers or the numbers of terms in the three functions. But for simplicity in this paper, it is assumed that the maximum powers of two variables in U and V functions are the same, because they are in-plane displacement components, or $N_U = N_V$ and $M_U = M_V$. The results are given in Table 1, which are obtained after fourteen iterative calculations.

To the first ten frequency parameters in the table, it is known that the lower frequency parameters converge fast than the higher ones and a good rate of convergence is obtained as the maximum powers of X

Table 1
Convergence of λ_i for various number of terms in displacement functions ($b_0/t = 25$, $b_0/l = 0.5$, $\alpha = 0.6$, $\beta = 60°$, $K = 60°$, $\phi = 45°$, $X_0 = 2.0$, $\bar{\Omega} = 1.0$)

N_U, M_U	7, 6	7, 7	7, 7	7, 7	8, 7	8, 7
N_V, M_V	7, 6	7, 7	7, 7	7, 7	8, 7	8, 7
N_W, M_W	8, 7	8, 7	8, 8	9, 8	8, 8	9, 8
Terms	49/49/56	56/56/56	56/56/63	56/56/72	64/64/63	64/64/72
1	9.4108	9.4936	9.4105	9.4090	9.4096	9.4083
2	27.947	28.379	27.946	27.941	27.940	27.936
3	47.252	47.538	47.251	47.243	47.242	47.236
4	71.986	73.067	71.982	71.978	71.963	71.950
5	101.41	102.98	101.41	101.40	101.32	101.30
6	117.18	117.73	117.17	117.15	116.95	116.91
7	136.51	136.96	136.48	136.44	135.81	135.72
8	157.74	157.92	157.69	157.64	157.04	156.90
9	189.93	187.83	189.88	189.72	187.15	186.80
10	205.67	203.96	205.64	205.57	203.21	202.89

Table 2
Comparisons of frequency parameters λ_i for rotating cylindrical thin panels ($b_0/t = 20$, $b_0/l = 0.5$, $\alpha = 1.0$, $\phi = 0°$, $X_0 = 0.0$, $\bar{\Omega} = 0.8$)

Method	$K = 0°$				$K = 60°$			
	$\beta = 30°$		$\beta = 60°$		$\beta = 30°$		$\beta = 60°$	
	Ref.	Present	Ref.	Present	Ref.	Present	Ref.	Present
1	6.2984	6.2977	9.6696	9.6677	4.8728	4.8865	6.2134	6.2030
2	15.393	15.393	15.560	15.562	17.230	17.258	19.393	19.391
3	32.141	32.139	49.260	49.254	31.351	31.246	36.144	35.951
4	49.324	49.324	50.770	50.770	52.177	52.178	54.311	54.339
5	57.471	57.457	57.628	57.614	75.390	75.175	76.140	75.843
6	78.307	78.303	90.201	90.200	87.821	87.787	91.338	91.266
7	92.972	92.972	100.06	100.05	94.268	94.135	105.91	105.35
8	95.100	95.098	108.40	108.39	109.40	108.99	109.19	109.02
9	125.51	125.50	126.99	126.98	131.37	131.06	135.39	134.72
10	140.89	140.31	167.51	166.05	144.15	143.58	161.17	159.62

Ref. denotes the reference (Hu and Tsuiji, 1999).

and θ are 8 and 7 in U and V functions, respectively, and 9 and 8 in W function, where the maximum error in the higher frequency parameters is <0.2%. Herein, those coefficients are utilized in the following analyses.

3.2. Comparison with previous method

As we known there were no researches on rotating twisted and open conical shells, so a rotating twisted cylindrical thin panels with the following parameters: $b_0/t = 20$, $b_0/l = 0.5$, $\alpha = 1.0$, $K = 0°$ and $60°$, $\beta = 30°$ and $60°$, a setting angle $\phi = 0°$, a radius $X_0 = 0.0$ and an angular velocity $\bar{\Omega} = 0.8$ is considered for demonstrating the practicability and the accuracy of the present method. The first ten frequency parameters obtained by the present method are compared with those in a reference (Hu and Tsuiji, 1999) shown in Table 2. It can be seen they have a good agreement because the maximum difference between them is <1%.

3.3. Effects on vibration characteristics

The effect of a subtended angle β for rotating twisted and open conical shells with various twist angles $K = 0°$, $30°$ and $60°$ on vibration frequency parameters λ is investigated, which is shown in Table 3. It is known that, for a given K, the frequency parameters increase when the β increases, because the curvature of conical shells increases as the β becomes large but the length of arcs does not change, which leads the stiffness of the shells to increase. The variations of λ are great for the lower modes and the lower symmetric modes, the maximum variation is corresponding to the first mode. In the case of $K = 0°$ when the β changes from 30° to 90°, the variations of the frequency parameters corresponding to the first (mode 1) and second (mode 3) bending vibration modes are 97.87% and 88.33%, respectively, but there are only 3.96% and 19.15% for the first (mode 2) and second (mode 4) torsional vibration modes, respectively, for instance. As the twist angle K increases the variations for the most of frequency parameters tend to decrease due to coupled vibration arising out of twist, especially for the lower ones. For an example, the variations of the first λ are 97.87%, 78.34% and 54.25% in the cases of $K = 0°$, $30°$ and $60°$ as the β varies from 30° to 90°.

Table 4 shows the variations of frequency parameters λ versus a tapered ratio of cross-section α for rotating open conical shells with $b_0/t = 25$, $b_0/l = 0.5$, $\beta = 60°$, $\bar{\Omega} = 1.0$, $X_0 = 1.0$ and $\phi = 0°$. For untwisted rotating conical shells, the frequency parameters decrease with the tapered ratio α increasing except those which represent the mode shapes of vibration varying with the change of the α, such as the third and

Table 3
Effects of β and K on frequency parameters λ_i ($b_0/t = 25$, $b_0/l = 0.5$, $\alpha = 0.6$, $\phi = 0°$, $X_0 = 1.0$, $\bar{\Omega} = 1.0$)

	$K = 0°$			$K = 30°$			$K = 60°$		
	$\beta = 30°$	$\beta = 60°$	$\beta = 90°$	$\beta = 30°$	$\beta = 60°$	$\beta = 90°$	$\beta = 30°$	$\beta = 60°$	$\beta = 90°$
1	9.0795	13.315	17.966	8.2540	11.249	14.720	7.2918	8.9381	11.249
2	23.912	24.320	24.860	27.106	28.316	29.102	22.057	27.169	30.491
3	36.641	56.698	69.005	33.711	46.995	58.722	43.970	47.352	53.747
4	64.574	66.456	76.938	68.149	72.823	77.593	62.177	71.158	76.783
5	78.995	79.468	80.391	82.226	87.134	92.181	100.02	101.44	103.63
6	86.035	122.25	129.92	89.641	113.79	127.95	102.20	116.83	129.18
7	117.06	122.58	141.10	122.07	130.79	142.31	127.47	135.50	145.75
8	149.76	154.74	164.41	150.80	155.82	165.00	154.87	156.77	163.95
9	154.53	192.93	206.46	154.22	188.47	198.58	163.91	186.93	190.32
10	183.89	196.98	215.57	187.03	198.83	219.97	193.92	202.56	221.54

Table 4
Effects of α and K on frequency parameters λ_i ($b_0/t = 25$, $b_0/l = 0.5$, $\beta = 60°$, $\phi = 0°$, $X_0 = 1.0$, $\bar{\Omega} = 1.0$)

	$K = 0°$			$K = 30°$			$K = 60°$		
	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 1.0$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 1.0$	$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 1.0$
1	15.273	13.315	12.497	13.604	11.249	9.9405	11.181	8.9381	7.8007
2	40.599	24.320	16.621	44.660	28.316	20.544	34.834	27.169	21.839
3	52.466	56.698	53.520	47.619	46.995	45.501	59.606	47.352	42.428
4	89.495	66.456	59.314	96.893	72.823	61.127	90.894	71.158	59.599
5	102.03	79.468	71.924	104.94	87.134	78.530	112.68	101.44	83.811
6	112.88	122.25	90.873	109.06	113.79	92.494	121.53	116.83	97.947
7	168.72	122.58	105.96	173.22	130.79	104.98	174.16	135.50	112.57
8	184.24	154.74	120.06	183.82	155.82	121.09	188.12	156.77	123.83
9	215.77	192.93	133.43	212.19	188.47	138.05	210.61	186.93	144.85
10	254.73	196.98	175.25	254.50	198.83	167.80	252.55	202.56	165.31

Table 5
Effects of ϕ and K on frequency parameters λ_i ($b_0/t = 25$, $b_0/l = 0.5$, $\alpha = 0.6$, $\beta = 60°$, $X_0 = 1.0$, $\bar{\Omega} = 1.0$)

	$K = 0°$			$K = 30°$			$K = 60°$		
	$\phi = 0°$	$\phi = 45°$	$\phi = 90°$	$\phi = 0°$	$\phi = 45°$	$\phi = 90°$	$\phi = 0°$	$\phi = 45°$	$\phi = 90°$
1	13.315	13.096	12.875	11.249	10.916	10.697	8.9381	8.4797	8.3977
2	24.320	24.128	23.913	28.316	28.051	28.313	27.169	27.153	27.150
3	56.698	56.657	56.650	46.995	46.963	46.815	47.352	47.328	47.755
4	66.456	66.399	66.309	72.823	72.789	72.887	71.158	71.200	71.183
5	79.468	79.513	79.534	87.134	86.939	87.695	101.44	101.56	102.03
6	122.25	122.19	122.19	113.79	113.74	113.90	116.83	116.94	117.38
7	122.58	122.63	122.62	130.79	130.73	130.96	135.50	135.51	135.69
8	154.74	154.69	154.65	155.82	155.74	155.83	156.77	156.80	156.98
9	192.93	192.93	192.92	188.47	188.38	188.72	186.93	187.02	187.49
10	196.98	197.02	197.05	198.83	198.86	199.00	202.56	202.73	202.82

sixth λ. For rotating twisted conical shells, the frequency parameters λ remain the tendency with the α and the tendency becomes stronger as the twist angle K increases.

The existence of a setting angle ϕ for a rotating twisted and open conical shell causes coupled vibrations, but there are symmetric and anti-symmetric mode shapes in both cases of $\phi = 0°$ and 90°. From Table 5, it can be seen that only the first frequency parameter λ shows the decrease monotonously as the ϕ increases for the conical shells with different twist angles, which will become large as the angular velocity increases. The effects on the others are complicated due to the coupled vibrations induced by the setting angle ϕ or the twist angle K.

The radius X_0 of a rotating disc is also an important parameter in a rotating system, Table 6 provides the relationship between the frequency parameters λ and the radius X_0 for given conical shells with and without twist. All the first ten λ increase with the increase of the X_0 for the rotating untwisted conical shell, and most of the λ remain the tendency for the rotating conical shell subjected twist deformation. For a given K, the variations of λ due to the radius X_0 are greater for the lower modes than for the higher ones, specially for the first λ. Also, it can be seen that the first λ decreases as the twist angle increases and the variation tends to decrease with the radius X_0 increasing, they are 59.29% and 33.26% in the cases of $X_0 = 0.0$ and 4.0, respectively, for instance.

Table 7 gives the effect of an angular velocity $\bar{\Omega}$ on the frequency parameters λ of twisted and open conical shells. It is known that the increase of the $\bar{\Omega}$ makes all the λ increase, because the centrifugal forces increase with the angular velocity $\bar{\Omega}$ increasing, which leads the initial deformation and the initial stress resultants to increase then the stiffness of the conical shells increases. Comparing with the frequency parameters of non-rotating twisted and open conical shells, the variations of λ increase greatly with the $\bar{\Omega}$ increasing, they are greater for twisted conical shells than for untwisted ones and also for the lower modes than for the higher ones.

Table 6
Effects of X_0 and K on frequency parameters λ_i ($b_0/t = 25$, $b_0/l = 0.5$, $\alpha = 0.6$, $\beta = 60°$, $\phi = 0°$, $\bar{\Omega} = 1.0$)

	$K = 0°$					$K = 60°$				
	$X_0 = 0.0$	$X_0 = 1.0$	$X_0 = 2.0$	$X_0 = 3.0$	$X_0 = 4.0$	$X_0 = 0.0$	$X_0 = 1.0$	$X_0 = 2.0$	$X_0 = 3.0$	$X_0 = 4.0$
1	12.644	13.315	13.953	14.563	15.146	7.9376	8.9381	9.8240	10.627	11.366
2	23.743	24.320	24.879	25.422	25.953	26.301	27.169	27.955	28.676	29.342
3	55.960	56.698	57.428	58.147	58.858	47.579	47.352	47.284	47.315	47.410
4	65.566	66.456	67.326	68.176	69.010	70.301	71.158	71.914	72.603	73.245
5	79.453	79.468	79.486	79.507	79.529	101.75	101.44	101.21	101.03	100.87
6	121.04	122.25	123.44	124.52	125.47	116.93	116.83	116.85	116.94	117.07
7	121.59	122.58	123.56	124.61	125.77	135.18	135.50	135.80	136.07	136.33
8	154.44	154.74	155.04	155.34	155.63	156.58	156.77	157.02	157.31	157.61
9	191.50	192.93	194.34	195.74	197.13	186.86	186.93	187.06	187.24	187.46
10	195.71	196.98	198.24	199.48	200.71	201.92	202.56	203.04	203.44	203.81

Table 7
Effects of $\bar{\Omega}$ and K on frequency parameters λ_i ($b_0/t = 25$, $b_0/l = 0.5$, $\alpha = 0.6$, $\beta = 60°$, $\phi = 0°$, $X_0 = 1.0$)

	$K = 0°$					$K = 60°$				
	$\bar{\Omega} = 0.0$	$\bar{\Omega} = 1.0$	$\bar{\Omega} = 2.0$	$\bar{\Omega} = 3.0$	$\bar{\Omega} = 4.0$	$\bar{\Omega} = 0.0$	$\bar{\Omega} = 1.0$	$\bar{\Omega} = 2.0$	$\bar{\Omega} = 3.0$	$\bar{\Omega} = 4.0$
1	12.123	13.315	16.361	20.411	24.952	7.1755	8.9381	12.716	17.095	21.670
2	23.165	24.320	27.450	31.926	37.221	25.641	27.169	30.363	33.442	36.623
3	55.453	56.698	60.292	65.864	72.968	48.265	47.352	47.480	50.703	56.710
4	64.882	66.456	70.854	76.794	78.110	69.549	71.158	74.177	77.484	81.269
5	79.516	79.468	79.376	79.903	86.563	102.47	101.44	99.895	98.550	99.137
6	120.18	122.25	127.24	134.52	143.87	117.47	116.83	116.83	119.39	126.25
7	120.97	122.58	128.17	137.33	148.99	134.90	135.50	136.57	138.53	143.11
8	154.20	154.74	156.34	158.93	162.47	156.79	156.77	158.01	160.37	163.60
9	190.53	192.93	199.91	211.00	219.02	187.17	189.93	187.32	190.00	197.11
10	194.91	196.98	202.95	211.88	225.57	201.24	202.56	203.82	206.24	212.41

The effect of the K on the first frequency parameter λ of rotating conical shells shows monotonous performance and the variations of λ with the K are little larger for lower modes than for higher ones. There is an obvious phenomenon for rotating conical shells subjected twist deformation that the number of iterative calculation increases rapidly until the first ten convergent frequency parameters λ can be obtained, which may be that the centrifugal force leads the twisted and open conical shells to deform against the twisted deformation.

4. Conclusions

A numerical method for free vibration analysis of rotating twisted and open conical shells is proposed based on the study of non-rotating twisted and open conical shells, where a non-linear strain–displacement relationship of the conical shells is adopted on the thin shell theory. There are two processes in this analysis method, first, the centrifugal force induced by rotating at a constant angular velocity is considered and the initial deformation and the initial stress resultants are obtained by the principle of virtual work for steady deformation. Second, under the conditions of the initial deformation and the initial stress resultants an energy equilibrium of equation for free vibration is presented by the principle of virtual work. The Rayleigh–Ritz procedure is used with two-dimensional polynomials as admissible displacement functions in the method. The effects of the parameters such as the twist angle, the subtended angle, the setting angle, the tapered ratio, and the radius and the angular velocity of a rotating disc on the vibration characteristics of conical shells are investigated.

It is known that the parameters of a rotating system have different influence on the vibration of rotating twisted and open conical shells. The subtended angle and the rotating radius increasing leads all of the frequency parameters to increase, there is an increasing tendency for the frequency parameters with the tapered ratio decreasing, the effect of the twist angle is complicated except the first frequency parameter shows monotonous performance, and the effects of the angular velocity and the setting angle also can be seen.

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Appendix A

The variables used in this paper are defined as the following,

$$f = a_{,x}\sin\theta - k(a\cos\theta - e),\quad h = a_{,x}\cos\theta - e_{,x} + ka\sin\theta,\quad p = f\sin\theta + h\cos\theta,\quad g = 1 + p^2,$$

$$q = f\cos\theta - h\sin\theta,\quad d_2 = p_{,x} + kp_{,\theta},\quad d_3 = ap_{,x} - p_{,\theta}q,\quad m = -\frac{1}{a\sqrt{g}}\left(1 - \frac{d_3}{g}\right). \tag{A.1}$$

Appendix B

The non-zero elements in matrix $\boldsymbol{G}$ are

$$G_{1,1} = 1,\quad G_{1,2} = -\frac{q}{\bar{a}},\quad G_{1,3} = \frac{p\bar{p}_{,x}}{g} + \frac{\bar{p}_{,x}}{g}\Phi_x^{(0)},\quad G_{1,6} = \frac{pp_{,\theta}}{\bar{a}g} + \frac{p_{,\theta}}{\bar{a}g}\Phi_x^{(0)},\quad G_{1,10} = \frac{1}{\sqrt{g}}\Phi_x^{(0)},$$

$$G_{1,11} = -\frac{q}{\bar{a}\sqrt{g}}\Phi_x^{(0)}, \quad G_{1,12} = -\frac{\bar{d}_3}{\bar{a}g\sqrt{g}}, \quad G_{2,1} = \frac{1}{\bar{a}\sqrt{g}}\left(1 - \frac{\bar{a}\bar{p}_{,x}}{g}\right), \quad G_{2,2} = \frac{1}{\bar{a}\sqrt{g}}\left(K + \frac{\bar{p}_{,x}q}{g}\right),$$

$$G_{2,3} = \frac{1}{\bar{a}g\sqrt{g}}\left(Ka_{,x}q + p\bar{p}_{,x} + q\bar{q}_{,x} + 2\frac{p\bar{p}_{,x}\bar{d}_3}{g}\right) - \frac{\bar{p}_{,x}\bar{m}}{g}\Phi_x^{(0)}, \quad G_{2,4} = -\frac{p_{,\theta}}{\bar{a}g\sqrt{g}}, \quad G_{2,5} = \frac{p_{,\theta}q}{\bar{a}^2g\sqrt{g}},$$

$$G_{2,6} = \frac{1}{\bar{a}^2g\sqrt{g}}\left(a_{,x}q - \bar{a}\bar{q}_{,x} + pp_{,\theta} + qq_{,\theta} + 2\frac{pp_{,\theta}\bar{d}_3}{g}\right) - \frac{p_{,\theta}\bar{m}}{\bar{a}g}\Phi_x^{(0)}, \quad G_{2,7} = -\frac{1}{g}, \quad G_{2,8} = -\frac{q^2}{\bar{a}^2g},$$

$$G_{2,9} = \frac{2q}{\bar{a}g}, \quad G_{2,10} = \frac{p\bar{d}_3}{\bar{a}g^2} - \frac{\bar{m}}{\sqrt{g}}\Phi_x^{(0)}, \quad G_{2,11} = -\frac{1}{\bar{a}^2g}\left(a_{,x}q - \bar{a}\bar{q}_{,x} + pp_{,\theta} + qq_{,\theta} + \frac{pq\bar{d}_3}{g}\right) + \frac{q\bar{m}}{\bar{a}\sqrt{g}}\Phi_x^{(0)},$$

$$G_{2,12} = -\frac{\bar{d}_2}{\bar{a}g^2}, \quad G_{3,4} = -\frac{p_{,\theta}}{\bar{a}^2g^2}, \quad G_{3,5} = -\frac{Kp_{,\theta}}{\bar{a}^2g^2}, \quad G_{3,7} = -\frac{1}{\bar{a}g\sqrt{g}}, \quad G_{3,8} = \frac{Kq}{\bar{a}^2g\sqrt{g}},$$

$$G_{3,9} = \frac{1}{\bar{a}^2g\sqrt{g}}(2q - p_{,\theta}), \quad G_{3,10} = \frac{p\bar{d}_2}{\bar{a}g^2\sqrt{g}}, \quad G_{3,11} = -\frac{1}{\bar{a}^3g\sqrt{g}}\left(a_{,x}q - \bar{a}\bar{q}_{,x} - K\bar{a}q_{,\theta} + \frac{\bar{a}pq\bar{d}_2}{g}\right),$$

$$G_{4,2} = \frac{q}{\bar{a}}, \quad G_{4,3} = \frac{a_{,x}}{\bar{a}} + \frac{K}{\sqrt{g}}\Phi_\theta^{(0)}, \quad G_{4,5} = \frac{1}{\bar{a}}, \quad G_{4,6} = -\frac{1}{\bar{a}\sqrt{g}}\Phi_\theta^{(0)}, \quad G_{4,11} = -\frac{1}{\bar{a}}\Phi_\theta^{(0)}, \quad G_{4,12} = \frac{1}{\bar{a}\sqrt{g}},$$

$$G_{5,1} = \frac{p_{,\theta}q}{\bar{a}g\sqrt{g}}, \quad G_{5,2} = -\frac{1}{\bar{a}\sqrt{g}}\left(K + \frac{\bar{p}_{,x}q}{g}\right), \quad G_{5,3} = \frac{1}{\bar{a}g\sqrt{g}}(p_{,\theta}\bar{q}_{,x} - p\bar{p}_{,x} - a_{,x}\bar{p}_{,x}) - \frac{K\bar{m}}{\sqrt{g}}\Phi_\theta^{(0)},$$

$$G_{5,4} = \frac{p_{,\theta}}{\bar{a}g\sqrt{g}}, \quad G_{5,5} = \frac{1}{\bar{a}^2\sqrt{g}}\left(1 - \frac{\bar{a}\bar{p}_{,x}}{g}\right), \quad G_{5,6} = -\frac{2pp_{,\theta}}{\bar{a}^2g\sqrt{g}} + \frac{\bar{m}}{\bar{a}\sqrt{g}}\Phi_\theta^{(0)}, \quad G_{5,8} = -\frac{1}{\bar{a}^2},$$

$$G_{5,10} = -\frac{p}{\bar{a}g}, \quad G_{5,11} = \frac{pq}{\bar{a}^2g} - \frac{\bar{m}}{\bar{a}}\Phi_\theta^{(0)}, \quad G_{5,12} = -\frac{\bar{d}_2}{\bar{a}g^2}, \quad G_{6,1} = -\frac{Kp_{,\theta}}{\bar{a}g^2}, \quad G_{6,4} = \frac{p_{,\theta}}{\bar{a}^2g^2},$$

$$G_{6,6} = \frac{p_{,\theta}}{\bar{a}^2g^2}\left(\frac{p\bar{d}_2}{g} - \frac{a_{,x}}{\bar{a}}\right), \quad G_{6,8} = \frac{\bar{p}_{,x}}{\bar{a}^2g\sqrt{g}}, \quad G_{6,9} = -\frac{p_{,\theta}}{\bar{a}^2g\sqrt{g}}, \quad G_{6,10} = \frac{p\bar{d}_2}{\bar{a}g^2\sqrt{g}}, \quad G_{6,11} = -\frac{pq\bar{d}_2}{\bar{a}^2g^2\sqrt{g}}$$

$$G_{7,1} = \frac{q}{\sqrt{g}}, \quad G_{7,2} = \frac{\sqrt{g}}{\bar{a}}\left(1 - \frac{q^2}{g}\right), \quad G_{7,3} = -\frac{1}{\bar{a}\sqrt{g}}\left(a_{,x}q - \bar{a}\bar{q}_{,x}\right) + \frac{\bar{p}_{,x}}{g}\Phi_\theta^{(0)} + \frac{K}{\sqrt{g}}\Phi_x^{(0)}, \quad G_{7,4} = \frac{1}{\sqrt{g}},$$

$$G_{7,5} = -\frac{q}{\bar{a}\sqrt{g}}, \quad G_{7,6} = -\frac{p}{\bar{a}\sqrt{g}} + \frac{\bar{p}_{,\theta}}{\bar{a}g}\Phi_\theta^{(0)} - \frac{1}{\bar{a}\sqrt{g}}\Phi_x^{(0)}, \quad G_{7,10} = \frac{1}{\sqrt{g}}\Phi_\theta^{(0)}, \quad G_{7,11} = \frac{1}{\bar{a}}\Phi_x^{(0)} - \frac{q}{\bar{a}\sqrt{g}}\Phi_\theta^{(0)},$$

$$G_{7,12} = -\frac{2p_{,\theta}}{\bar{a}g}, \quad G_{8,1} = \frac{2q}{\bar{a}g}, \quad G_{8,2} = \frac{2}{\bar{a}g}(Kq - \bar{p}_{,x}), \quad G_{8,3} = \frac{4p\bar{p}_{,x}p_{,\theta}}{\bar{a}g^2} - \bar{m}\left(\frac{\bar{p}_{,x}}{g}\Phi_\theta^{(0)} + \frac{K}{\sqrt{g}}\Phi_x^{(0)}\right),$$

$$G_{8,4} = \frac{2}{\bar{a}g}, \quad G_{8,5} = -\frac{2q}{\bar{a}^2 g}, \quad G_{8,6} = \frac{2}{\bar{a}^2 g}\left(\frac{2pp_{,\theta}^2}{g} - a_{,x}\right) - \bar{m}\left(\frac{p_{,\theta}}{\bar{a}g}\Phi_\theta^{(0)} - \frac{1}{\bar{a}\sqrt{g}}\Phi_x^{(0)}\right), \quad G_{8,8} = \frac{2q}{\bar{a}^2\sqrt{g}},$$

$$G_{8,9} = -\frac{2}{\bar{a}\sqrt{g}}, \quad G_{8,10} = \frac{2pp_{,\theta}}{\bar{a}g\sqrt{g}} - \frac{\bar{m}}{\sqrt{g}}\Phi_\theta^{(0)}, \quad G_{8,11} = \frac{2}{\bar{a}^2\sqrt{g}}\left(a_{,x} - \frac{pp_{,\theta}q}{g}\right) - \bar{m}\left(\frac{1}{\bar{a}}\Phi_x^{(0)} - \frac{q}{\bar{a}g}\Phi_\theta^{(0)}\right),$$

$$G_{9,7} = -\frac{p_{,\theta}}{\bar{a}g^2}, \quad G_{9,8} = -\frac{1}{\bar{a}^2 g}\left(K + \frac{\bar{p}_{,x}q}{g}\right), \quad G_{9,9} = -\frac{1}{\bar{a}^2 g}\left[1 - \frac{1}{g}\left(\bar{a}\bar{p}_{,x} + p_{,\theta}q\right)\right],$$

$$G_{9,11} = \frac{1}{\bar{a}^2 g}\left[\frac{a_{,x}}{\bar{a}} + \frac{1}{g}\left(p_{,\theta}\bar{q}_{,x} - \frac{a_{,x}p_{,\theta}q}{\bar{a}} - a_{,x}\bar{p}_{,x} - Kpp_{,\theta}\right)\right]. \tag{B.1}$$

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